of airwall structures," Goodyear Aircraft Corp., GER5850 (February 19, 1954).

[4] Harris, J. T. and Stimler, F. J., "Expandable structures for space," Astronautics **6**, 30 (April 1961).

[5] Darevskii, V. M., "Stability of a console cylindrical shell under bending by a transverse force with twisting and internal pressure," ARS J. **31**, 125–133 (1961).

[6] Hopkins, H. G. and Brown, E. H., "The effect of internal pressure on the initial buckling of thin-walled circular cylinders under torsion," Ames Research Center, Rept. and Memo. 2423 (1951).

[7] Peterson, J. P., "Correlation of the buckling strength of pressurized cylinders in compression or bending with structural parameters," NASA TN D-526 (October 1960).

[8] Harris, L. A., Suer, S. H., Skene, W. T., and Benjamin, R. J., "The stability of thin-walled unstiffened circular cylinders under axial compression including the effects of internal pressure," J. Aeronaut. Sci. **24**, 587–596 (1957).

[9] Lo, H., Crate, H., and Schwartz, E. B., "Buckling of thin-walled cylinder under axial compression and internal pressure," NACA TN 1027 (1951).

[10] Fung, V. C. and Sechler, E. E., "Buckling of thin-walled circular cylinders under axial compression and internal pressure," J. Aeronaut. Sci. **24**, 351–356 (1957).

Minimum Altitude Variation Orbits about an Oblate Planet

FORD KALIL*
The Martin Company, Baltimore, Md.

Introduction

THE general problem of satellite motion over an oblate earth has been studied at length, and the literature is too extensive to list here. The study used in this work is that of Struble.[1] In addition, Rider[2] has studied a "Class of Minimum Altitude Variation Orbits About an Oblate Earth," in which case the resulting orbits were essentially of zero eccentricity. Rider uses the results obtained by Fosdick[3] for the motion of a near-earth satellite. However, the problem of maintaining a minimum satellite altitude variation during certain phases of satellite motion and for a short orbital life has not been reported in the literature.

A satellite orbit that closely follows the contour of the oblate earth during one-quarter of a revolution could be useful, first, in studying the flattening of the atmosphere, which is usually assumed to have the same flattening as the earth because density variations can be measured more accurately than the absolute value. Hence, a null method of measurement for verifying this assumption probably could be used to advantage. Secondly, such an orbit could be useful in photographic missions because the camera's resolution is best at the altitude for which the optical system is preset, and thus the degradation of the resolution is minimized when the altitude variations are minimized.

The purpose of this study is to derive a method and corresponding analytical expressions for determining the orbital parameters for an orbit that closely follows the contour of the earth during one-quarter of each revolution with a minimum altitude variation throughout an operational time of several days. In this work, the first-order theory of Struble was used. A cursory study indicates that such a class of orbits is quite feasible.

The effects of atmospheric drag decay by an oblate, rotating atmosphere on orbits whose $0 \leq e \leq 0.01$ have been studied.[4–10] For an $\bar{h} \simeq 100$ naut miles, $e \simeq 0.0022$ (the e for minimum altitude variation over one-quarter of each orbit revolution), and $B = (C_D A/2m) = 1$, one has $\Delta a/\text{rev} \cong 0.6$ naut mile per revolution, and $\Delta e/\text{rev} \cong 3 \times 10^{-7}$. Since the actual ballistic coefficient B is generally significantly less than 1, it safely can be said that, for short-term missions, the effect of drag is negligible for all practical purposes.

Received by ARS October 2, 1962; revision received March 29, 1963.

* Design Engineer, Space Systems Division.

Theory

The earth's local radius R is given to within about 92 ft by

$$R = R_e(1 - f\sin^2\theta) \tag{1}$$

where

$$\sin^2\theta = \sin^2 i_0 \sin^2\beta \tag{2}$$

and θ = geocentric latitude, $f = 1/298.25$ = earth's flattening, and R_e = earth's equatorial radius.

The local satellite altitude h_β above the earth's surface is given by

$$h_\beta = r - R \tag{3}$$

where r is the local satellite orbital radius, measured from the earth's center. For a Keplerian orbit, r is given by

$$r = \frac{a(1 - e^2)}{1 + e\cos(\beta - \omega)} \tag{4}$$

where a is the semimajor axis an l e is the eccentricity.

Since there was no a priori knowledge of the magnitude of e, the Keplerian orbit was examined first. It was found that $e = 0(J)$ for orbits in which the satellite is to follow the contour of the earth during one-quarter of a revolution, making it permissible to drop terms involving e^2 or Je in the first-order theory.

Using Struble's results, the satellite radial position from the earth's center is given by

$$1/r = (1/\bar{r}_0)[1 + e\cos(\beta - \omega) - Jv] \tag{5}$$

where $\bar{r}_0$ and e are arbitrary constants, and

$$v = \frac{1}{12}\left(\frac{R_e}{\bar{r}_0}\right)^2 \sin^2 i_0 \left[\left(2 + \frac{e^2}{3}\right)\cos 2\beta + e\cos(3\beta - \omega) + \frac{e^2}{6}\cos(4\beta - 2\omega) + \frac{3e^2}{2}\cos 2\omega\right] + \frac{1}{12}\left(\frac{R_e}{\bar{r}_0}\right)^2 e^2(2 - 3\sin^2 i_0)\cos(2\beta - 2\omega) \tag{6}$$

$$d\omega/d\beta = (J/2)(R_e/\bar{r}_0)^2(5\cos^2 i_0 - 1) \tag{7}$$

The instantaneous line of nodes regresses at a variable rate, and to first order is given by

$$\Omega = J(R_e/\bar{r}_0)^2 \cos i_0[\beta - \tfrac{1}{2}\sin 2\beta + e\sin(\beta - \omega) - \tfrac{1}{6}e\sin(3\beta - \omega) - \tfrac{1}{2}e\sin(\beta + \omega)] \tag{8}$$

From Eqs. (7) and (8), the secular variations in ω and Ω, respectively, are given by

$$\dot{\omega}\frac{\text{deg}}{\text{rev}} = 360\left(\frac{J}{2}\right)\left(\frac{R_e}{\bar{r}_0}\right)^2(5\cos i_0 - 1) \tag{9}$$

$$\dot{\Omega}\frac{\text{deg}}{\text{rev}} = -360J\left(\frac{R_e}{\bar{r}_0}\right)^2\cos i_0 \tag{10}$$

where a satellite revolution is defined as extending from one ascending node to the next.

In these expressions, ω is the argument of perigee, β is the central angle in the orbital plane measured from the mean line of nodes, i_0 is the mean angle of incidence of the orbital plane, and $J = 1.6235 \times 10^{-3}$ is the coupling coefficient of the second harmonic in the earth's gravitational potential.

Since J is of the order of 10^{-3}, then for small e (i.e., e of the order of 10^{-3}), terms involving e^2 or Je may be neglected, in which case

$$h_\beta = \bar{h} - r_0 e \cos(\beta - \omega) - B \cos 2\beta \tag{11}$$

In addition, let

$$\Delta h = h_\beta - \bar{h} = -\bar{r}_0 e \cos(\beta - \omega) - B \cos 2\beta \tag{12}$$

where

$$\bar{h} = \bar{r}_0 - R_e[1 - (f/2) \sin^2 i_0] \tag{13}$$

$$B = [(R_e f/2) - (J/6)(R_e^2/\bar{r}_0)] \sin^2 i_0 \tag{14}$$

The physical significance of $\bar{h}$ is that it is the average altitude about which h_β varies periodically during a satellite revolution, keeping in mind that h_β is the satellite altitude above the surface of an oblate earth. Hence, Eq. (11) is a very elucidating equation. For example, if the two terms involving β are plotted as functions of β for a complete revolution, $0° \leq \beta \leq 360°$ at $\omega = 0°$, then for minimum altitude variation for $0° \leq \beta \leq 90°$ (i.e., during the ascending phase of satellite motion over the northern hemisphere), it can be seen that the two terms must be symmetrically out of phase for optimum cancellation. This is done readily by setting $\omega = 135°$ so that Δh at $\beta = 45°$ is zero.

Similarly, for a minimum altitude variation for $90° \leq \beta \leq 180°$ (i.e., during the descending phase of satellite motion over the northern hemisphere), the two terms are symmetrically out of phase when $\omega = 45°$. In this case Δh at $\beta = 135°$ is zero. Mathematically, this can be expressed as follows: for the two periodic terms in the Δh of Eq. (12) to be symmetrically out of phase,

$$\Delta h = 0 \quad \text{at} \quad \beta = (\pi/4) + (q - 1)(\pi/2) \tag{15}$$

in which case the optimum $\omega = \bar{\omega}$ is given by

$$\begin{aligned} \omega &= (3\pi/4) - (q - 1)(\pi/2) \\ \text{rad} &= 135° - (q - 1)90° \end{aligned} \tag{16}$$

where q is the quadrant of the central angle β during which it is desired to keep Δh at a minimum.

Consider now $\Delta h = h_\beta - \bar{h}$, which is the altitude variation of h_β about $\bar{h}$. Equation (12) for $\omega = 135°$ becomes

$$\Delta h = (\bar{r}_0 e/2^{1/2})(\cos\beta - \sin\beta) - B \cos 2\beta \quad \text{at } \omega = 135° \tag{17}$$

and it can be seen by referring to Fig. 1 that one condition for minimum Δh is given by

$$\Delta h(\beta = 0) = -\Delta h(\beta = \beta') \qquad \text{for} \begin{cases} \Delta h = \min \\ 0° \leq \beta \leq 90° \end{cases} \tag{18}$$

where β' is the central angle at which the slope of Δh first becomes zero. Solving Eqs. (17) and (18) at $\beta = 0$, $\beta = \beta'$, the following is obtained:

$$\frac{\bar{r}_0 e}{2^{1/2} B} = \frac{1 + \cos 2\beta'}{1 + \cos\beta' - \sin\beta'} \qquad \text{for} \begin{cases} \Delta h = \min \\ 0° \leq \beta \leq 90° \end{cases} \tag{19}$$

Since the slope of Δh is zero at $\beta = \beta'$, then

$$(\partial \Delta h/\partial \beta)_{\beta = \beta'} = -(\bar{r}_0 e/2^{1/2})(\cos\beta' + \sin\beta') + 2B \sin 2\beta' = 0 \tag{20}$$

Solving Eqs. (19) and (20) for β', then, the following expression results:

$$1 - \cos\beta' \sin\beta' + \sin^2\beta' - 2 \sin\beta' = 0 \qquad \text{for} \begin{cases} \Delta h = \min \\ 0° \leq \beta \leq 90° \end{cases} \tag{21}$$

from which

$$\beta' = 23.3° \tag{22}$$

It is interesting to note that, once the region over which Δh is to be minimum is specified ($0° \leq \beta \leq 90°$ in this case), then ω is specified automatically, and β' also is specified automatically by Eq. (21) because it is independent of all other parameters. Then, the optimum value of eccentricity [Eq. (19)] becomes

$$e = 1.56667 \frac{B}{\bar{r}_0} = \frac{91}{60} \frac{B}{\bar{r}_0} \qquad \text{for} \begin{cases} \Delta h = \min \\ 0° \leq \beta \leq 90° \end{cases} \tag{23}$$

Consider now the regression of the angle of perigee ω in the orbital plane. If it is desired to maintain a minimum altitude variation for the qth quadrant over a period of n satellite revolutions, then choose

$$\omega_0 = \bar{\omega} - n(\dot{\omega}/2) \qquad \text{at the start of the mission} \tag{24}$$

$$\omega = \bar{\omega} \qquad \text{at } n/2 \tag{25}$$

$$\omega_f = \bar{\omega} + n(\dot{\omega}/2) \qquad \text{at the end of the mission} \tag{26}$$

Results

From the symmetry of the earth, some generalizations can be made and the theoretical results summarized as follows. Specify $\bar{h}$, i_0, and the quadrant q of the central angle β during which the altitude variation is to be minimum and the total number of revolutions n is required for the satellite operation.

In general, to first order,

$$\bar{r}_0 = \bar{h} + R_e[1 - (f/2) \sin^2 i_0] \tag{27}$$

$$B = [(R_e f/2) - (J/6)(R_e^2/\bar{r}_0)] \sin^2 i_0 \tag{28}$$

$$r = \bar{r}_0[1 - e \cos(\beta - \omega) + (J/6)(R_e/\bar{r}_0)^2 \sin^2 i_0 \cos 2\beta] \quad e = 0(J) \tag{29}$$

$$r = h_\beta + R_e(1 - f \sin^2 i_0 \sin^2\beta) \tag{30}$$

$$h_\beta = \bar{h} - \bar{r}_0 e \cos(\beta - \omega) - B \cos 2\beta \tag{31}$$

$$\Delta h = h_\beta - \bar{h} \tag{32}$$

For minimum altitude variation during a quarter of a revolution (i.e., in the quadrant q),

$$\bar{\omega} = 135° - (q - 1)90° \tag{33}$$

$$e = \tfrac{91}{60}(B/\bar{r}_0) \tag{34}$$

Set

$$\omega_0 = \bar{\omega} - (\Delta\omega/2) \qquad \text{at the start of the mission} \tag{35}$$

$$\omega = \bar{\omega} \qquad \text{at } n/2 \tag{36}$$

$$\omega_f = \bar{\omega} + (\Delta\omega/2) \qquad \text{at the end of the mission} \tag{37}$$

where

$$\Delta\omega = 180\, n\, J(R_e/\bar{r}_0)^2(5 \cos^2 i_0 - 1) \tag{38}$$

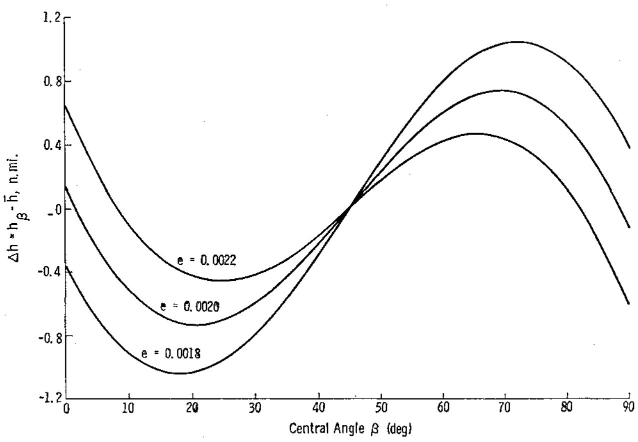

Fig. 1 Altitude variation vs β for various eccentricities at $\omega = \bar{\omega} = 135°$, $i_0 = 90°$, and $\bar{h} = 100$ naut miles

in which case $\Delta h \cong \pm 0.5$ naut mile in the qth quadrant for low-altitude orbits.

It is interesting to note that when $i_0 = 0$, both B and e become zero, and hence $\Delta h = 0$, which says that, for an equatorial orbit, a "circular" orbit gives the minimum altitude variation. In addition, a polar orbit gives the greatest altitude variation.

References

[1] Struble, R. A. and Campbell, W. F., "Theory of motion of a near earth satellite," ARS J. **31**, 154–155 (1961).

[2] Rider, L., "A class of minimum altitude variation orbits about an oblate earth," Aerospace Corp., El Segundo, Calif., TDR-594 (1560-01) TN-2 (March 1961); also ARS J. **31**, 1580–1582 (1961).

[3] Anthony, M. L. and Fosdick, G. E., "Satellite motions about an oblate planet," J. Aerospace Sci. **28**, 789–802 (1961).

[4] Kalil, F., "Effect of an oblate rotating atmosphere on the eccentricity, semimajor axis, and period of a close earth satellite," The Martin Co., ER 12552 (April 1962) and ER 12511 (May 1962); also AIAA J. (to be published).

[5] Sterne, T. E., "Effect of the rotation of a planetary atmosphere upon the orbit of a close satellite," ARS J. **29**, 777–782 (1959).

[6] Brouwer, D. and Hori, G., "Theoretical evaluation of atmosphere drag effects in the motion of an artificial satellite," Astron. J. **66**, no. 5, 193–225 (1961).

[7] Brouwer, D. and Hori, G., "Appendix to theoretical evaluation of atmospheric drag effects in the motion of an artificial satellite," Astron. J. **66**, no. 6, 264–265 (1961).

[8] King-Hele, D. G., "The effect of the earth's oblateness on the orbit of a near satellite," Proc. Roy. Soc. (London) **247A**, 49–72 (August 1958).

[9] King-Hele, D. G., Cook, G. E., and Walker, D. M. C., "Contraction of satellite orbits under the influence of air drag, Part I, with a spherically symmetrical atmosphere," Royal Aircraft Establishment, Farnborough, England, TN G.W. 533 (November 1959).

[10] King-Hele, D. G., Cook, G. E., and Walker, D. M. C., "The contraction of satellite orbits under the influence of air drag, Part II, with oblate atmosphere," Royal Aircraft Establishment, Farnborough, England, TN G.W. 565 (December 1960).

Cryogenic Propellant Stratification Analysis and Test Data Correlation

T. Bailey,* R. VandeKoppel,† G. Skartvedt‡
Martin Company, Denver, Colo.

AND

T. Jefferson§
University of Arkansas, Fayetteville, Ark.

Nomenclature

A_H = tank wall heated area
A_{HC} = heated area opposite cold layer
C_p = liquid specific heat
h = liquid film heat-transfer coefficient
q = heat-transfer rate through unit area of tank wall
u = velocity of fluid in boundary layer
V_H = volume of warm layer
y = distance in boundary layer measured from wall
δ = boundary-layer thickness

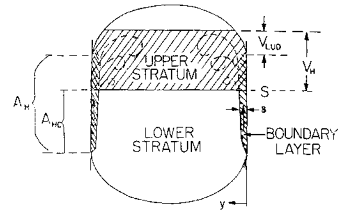

Fig. 1 Analytical model

θ = temperature excess (actual temperature minus liquid-bulk temperature)
θ_w = wall temperature minus liquid-bulk temperature
ρ = liquid density
H = liquid enthalpy per unit mass
D = tank diameter

One of the primary objectives of a liquid rocket propellant pressurization and feed system is to supply propellant to the engine at the required pressure and flow rate. For a pump-fed engine, sufficient pressure is required to suppress cavitation in the pump. This pressure usually is specified in terms of a minimum net positive suction head (NPSH) value, with NPSH defined as the difference between total pressure and vapor pressure at the pump inlet. The rocket designer must provide the required margin between total pressure and vapor pressure at all times during flight. Design of cryogenic propellant systems in particular requires detailed analysis of transient temperature distributions within the propellant resulting from aeroheating. A variety of test data is available which indicates a strong tendency toward stratification; the upper propellant layers experience a large increase in vapor pressure, and the lower layers show little or none. The upper layers may or may not be saturated, depending on the system operating parameters.

An analytical procedure has been developed to accomplish quantitative analysis of the stratification phenomenon. Development of the analysis proceeds from basic considerations, and no scale-effect factors or other gross empirical coefficients are required. Results of the analysis consist of volume and temperature of the warm upper propellant layer as functions of time. Correlations of predicted results with liquid nitrogen ground-test data and liquid oxygen Titan and Vanguard flight-test data confirm the validity and usefulness of the analytical model.

Analytical Model

The stratification analysis is based on integration of liquid mass flow in the natural convection boundary layer along the heated tank wall. The primary assumptions of the analysis are that 1) the initial temperature of the liquid is uniform, 2) all the heat input to the tank wall appears as sensible heat in the boundary layer, 3) all the flow in this boundary layer goes into a warm upper stratum and remains there, 4) this warm stratum is uniformly mixed, and 5) there is no mixing between the warm stratum and the lower unheated stratum.

Consider the horizontal plane S separating the two strata, as shown in Fig. 1. The growth of the upper stratum results from the flow in the boundary layer that crosses S. This flow is confined to the annular ring in S, of width δ, inside the tank wall. Applying an energy balance to that portion of the boundary layer below S and assuming that the thermal energy stored in the boundary layer is negligible, there results the equation

$$q\, A_{HC} = \pi D \int_0^\delta \rho [H(y) - H_0]\, u(y)\, dy \tag{1}$$

Assuming constant specific heat and density, this can be written as

$$q\, A_{HC} = C_p \rho \pi D \int_0^\delta \theta(y)\, u(y)\, dy \tag{2}$$

Received by ARS June 29, 1962; revision received April 17, 1963.

* Design Engineer.
† Design Specialist. Member AIAA.
‡ Unit Head.
§ Head, Mechanical Engineering Department.